Answers to Second Midterm

MA441: Algebraic Structures I

14 December 2003

All questions are worth ten points. In addition to the questions, there will be an additional ten points to be awarded for style and clarity in writing.

1) Definitions

1. Define the **index** $|G : H|$ of a subgroup $H < G$.

   The index is equal to the number of distinct left cosets of H in G. The index is also equal to $|G|/|H|$.

2. Define the **external direct product** $G_1 \oplus G_2$ of groups G_1 and G_2. (Specify the set of elements that make up the direct product and its group operation.)

   The external direct product consists of the set of pairs $\{(a, b)|a \in G_1, b \in G_2\}$ with the group operation $(a, b) \cdot (c, d) = (ab, cd)$, where ac is composed in G_1 and bd is composed in G_2.

2) Fill in the blanks or answer True/False.

1. $11^{29} \equiv 11 \pmod{29}$.

2. True or False: Every group G of order 7 contains an element $a \in G$ such that $|a| = 7$. True

3. True or False: if $\mathrm{Aut}(G_1) \approx \mathrm{Aut}(G_2)$ then $G_1 \approx G_2$. False

4. In S_5, $|(12)(13)(456)| = 3$ (Compose left to right.)

5. True or False: Let G be an arbitrary finite group of order n. If $d|n$, then there is an $H < G$ of order d. False

3) Lagrange's Theorem

1. Give a clear and complete statement of Lagrange's Theorem for a subgroup H of a finite group G.

   Let H be a subgroup of G. Then the order of H divides the order of G. The number of cosets of H in G is $|G|/|H|$.

2. Use Lagrange's Theorem to prove that the order of an element $a \in G$ divides the order of G.

   The cyclic subgroup $\langle a \rangle$ is a subgroup of G, so its order $|\langle a \rangle|$ divides the order of G. The order of a, $|a|$ equals $|\langle a \rangle|$, so $|a|$ divides $|G|$.

3. Use Lagrange's Theorem to prove that if $|G|$ is prime, then G must be cyclic.

   If $|G|$ is prime, then the order of any element is either 1 or $|G|$. Therefore any nonidentity element has order $|G|$ and generates G.

4. If H and K are subgroups of G, where $|H| = 33$ and $|K| = 28$, what are the possible orders of $H \cap K$? Explain your answer.

   Since the intersection $H \cap K$ is a subgroup of both H and K, its order must divide both 33 and 28 by Lagrange's Theorem. However, $33 = 3 \cdot 11$ and $28 = 2^2 \cdot 7$ are relatively prime, so the order of the intersection must be 1.

4) List the distinct left cosets of H in G if

1. $G = D_4$, $H = \langle FR \rangle$, where R is a rotation and F is a flip.

   The distinct left cosets can be written as $H = \{e, FR\}$, $RH = \{R, F\}$, $R^2H = \{R^2, FR^3\}$, $R^3H = \{R^3, FR^2\}$.

2. $G = \mathbb{Z}/15\mathbb{Z}$, $H = \langle 3 \rangle$.

   $H = \{0, 3, 6, 9, 12\}$, $1 + H = \{1, 4, 7, 10, 13\}$, $2 + H = \{2, 5, 8, 11, 14\}$.

5) Suppose $\phi : G_1 \to G_2$ is an isomorphism. Prove that G_1 is abelian if and only if G_2 is abelian.

Suppose G_1 is abelian. Any pair of elements $x, y \in G_2$ are the images of some elements $a, b \in G_1$. Let's say $\phi(a) = x$ and $\phi(b) = y$. Then since G_1 is abelian,

$$xy = \phi(a)\phi(b) = \phi(ab) = \phi(ba) = \phi(b)\phi(a) = yx.$$

Therefore G_2 is abelian.

Suppose G_2 is abelian. Then $\phi(ab) = \phi(a)\phi(b) = xy$. Also, $\phi(ba) = \phi(b)\phi(a) = yx$. Since G_2 is abelian, $xy = yx$. Because ϕ is one-to-one, since ab and ba have the same image, then $ab = ba$. Therefore G_1 is abelian.

6) Cosets: Given a subgroup $H < G$, prove that any two cosets of H have the same order, that is, for any $a, b \in G$, $|aH| = |bH|$.

Let $\psi : aH \to bH$ via $ah \mapsto bh$ $(\forall h \in H)$. The map ψ is one-to-one because if $ah = bh$ then $a = b$ because we can right multiply by h^{-1}. The map ψ is onto because any element in the coset bH has the form bh, which has preimage ah under ψ. This shows that ψ is a bijection on the two cosets aH and bH, which shows $|aH| = |bH|$.

7) Classification: Prove the following:
Let G be a group of order $2p$, for $p > 2$ prime. Suppose G is not cyclic and that $a \in G$ has order p. If $b \notin \langle a \rangle$, show that $|b| = 2$.

Since $|a| = p$, the index of $\langle a \rangle$ in G is 2, that is, there are exactly two cosets of $\langle a \rangle$ in G. The left coset $b\langle a \rangle$ must not be the same as $\langle a \rangle$ because then $b \in \langle a \rangle$. So the only two cosets are $\langle a \rangle$ and $b\langle a \rangle$.

The left coset $b^2\langle a \rangle$ must be one of these two cosets. If $b^2\langle a \rangle = b\langle a \rangle$, then by cancelling b on the left, we would get that $b \in \langle a \rangle$, so we eliminate that possibility. Therefore, $b^2\langle a \rangle = \langle a \rangle$, so $b^2 \in \langle a \rangle$.

Since $|\langle a \rangle| = p$, the order of b^2 must divide p, which means it is either 1 or p. If $|b^2| = p$, then $\langle b^2 \rangle = \langle a \rangle$. Since $b \notin \langle a \rangle$, then $\langle b \rangle$ must be all of G. But we assumed that G is not cyclic. Therefore, $|b^2|$ must be 1 and $|b| = 2$.

(This is part of Theorem 7.2.)

8) Let $|x| = 40$. List all the elements of $\langle x \rangle$ that have order 10. Explain your answer.

This is Exercise 46 from Chapter 4, taken from Homework Assignment 6.

The order of x^k is $40/\gcd(40, k)$. To get an element of order 10, choose k such that $\gcd(40, k) = 4$. The elements of order 10 are $\{x^4, x^{12}, x^{28}, x^{36}\}$.

9) (extra credit) Prove Lagrange's Theorem. You may cite basic properties of cosets, such as those listed in Gallian's Lemma, if you state them accurately.

Please refer to the proof on page 137 of Gallian. It suffices to note that G can be partitioned by its cosets, that all cosets have the same size, and that therefore the order of G is an even multiple of the order of a coset.