Second Midterm: Preparation

MA441: Algebraic Structures I

16 November 2003

The midterm will be composed of questions listed in this preparation. While some numbers may change, the questions will remain essentially the same. In addition to the questions listed below, there will be an additional ten points to be awarded for style and clarity in writing. This will be largely a judgment call on my part, however I will definitely deduct points for grammatical mistakes or handwriting that is hard to read.

There may be an extra credit problem on more recent material from Chapters 7–9 that will be worth up to 10 points.

1) Definitions (two from this list)

1. Define what it means for a permutation to be **odd**.
2. Define the **index** $|G : H|$ of a subgroup $H < G$.
3. Define the **alternating group** A_n.
4. Given a map ϕ from a group G_1 to another group G_2, define what it means for ϕ to be an **isomorphism**.
5. Define the **left coset** aH of a subgroup $H < G$.
6. Given a group $G < S_n$ that acts on a set $S = \{1, \ldots, n\}$, define the **stabilizer** $\mathrm{Stab}_G(i)$ of i in G.
7. Given a group $G < S_n$ that acts on a set $S = \{1, \ldots, n\}$, define the **orbit** $\mathrm{Orb}_G(i)$ of i under G.
8. Define the **external direct product** $G_1 \oplus G_2$ of groups G_1 and G_2.

2) Fill in the blanks or answer True/False (five from this list)

1. The order of the element $(2, 2)$ in $\mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$ is __.
2. True or False: $24 + 7\mathbb{Z} = 10 + 7\mathbb{Z}$. __
3. $17^{23} \equiv$ __ (mod 23).
4. If G is the group of nonzero real numbers under multiplication and H is the subgroup of positive reals, then $|G : H| =$ __.
5. True or False: the subgroup $2\mathbb{Z}$ has infinite index in $\mathbb{Z}$. __
6. True or False: D_4 is isomorphic to $\mathrm{Aut}(\mathbb{Z}/8\mathbb{Z})$. __
7. True or False: Every group G of order 6 contains an element $a \in G$ such that $|a| = 6$. __
8. $\mathrm{Aut}(\mathbb{Z}/12\mathbb{Z}) \approx U($__$)$.
9. True or False: if $\mathrm{Aut}(G_1) \approx \mathrm{Aut}(G_2)$ then $G_1 \approx G_2$.
10. In S_5, $|(12)(13)(45)| =$ __. (Compose left to right.)
11. True or False: (1234) is an even permuation. __
12. True or False: Let G be a cyclic group of order n. If $d|n$, then there is an $H < G$ of order d.
13. Let G be a cyclic group of order 7777. The number of elements of order 7 is __.

3) Lagrange's Theorem

1. Give a clear and complete statement of Lagrange's Theorem for a subgroup $H < G$. Explain how the orders of H and G are related and how many cosets there are of H in G.
2. Use Lagrange's Theorem to prove that the order of an element $a \in G$ divides the order of G.
3. Use Lagrange's Theorem to prove that if $|G|$ is prime, then G must be cyclic.

4. If H and K are subgroups of G, where $|H| = 30$ and $|K| = 22$, what are the possible orders of $H \cap K$? Explain your answer.

4) List the distinct cosets of H in G if (two of four)

1. $G = D_3$, $H = \langle FR \rangle$, where R is a rotation and F is a flip.
2. $G = \mathbb{Z}/12\mathbb{Z}$, $H = \langle 3 \rangle$.
3. $G = S_6$, $H = A_6$.
4. $G = U(12)$, $H = \langle 5 \rangle$.

5) Suppose $\phi : G_1 \to G_2$ is an isomorphism. Prove that (one of two)

1. G_1 is abelian iff G_2 is abelian.
2. If $H < G_1$, then $\phi(H) < G_2$.

6) Cosets. Given a subgroup $H < G$, prove that for any $a, b \in G$ (one of three)

1. $aH = bH$ iff $a^{-1}b \in H$.
2. $|aH| = |bH|$.
3. $aH = Ha$ iff $H = aHa^{-1}$

7) Classification. Prove (one of three)

1. If G is a group of order 4, prove that either $G \approx \mathbb{Z}/4\mathbb{Z}$ or $G \approx \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.
2. If G is a group of order at least three for which all non-identity elements have order 2, then G has a subgroup of order 4.
3. Let G be a group of order $2p$, for $p > 2$ prime. Suppose G is not cyclic and that $a \in G$ has order p. If $b \notin \langle a \rangle$, show that $|b| = 2$.

8) A homework problem chosen from assignments 4–8.