Final Exam Preparation

MA441: Algebraic Structures I

15 December 2003

All questions are worth ten points, unless otherwise indicated. In addition to the questions, there will be an additional ten points to be awarded for style and clarity in writing. The problems that appear on the actual exam may be slightly altered from the ones that appear here. There may also be an extra credit problem.

1) Definitions (four from this list, worth 20 points)

1. Define a group.
2. Given a group G, define what it means for G to be cyclic.
3. Define the order $|a|$ of an element $a \in G$.
4. Given a permutation $\alpha \in S_n$, define what it means for α to be an odd permutation.
5. Define the index $|G : H|$ of a subgroup $H < G$.
6. Given a subgroup $H < G$, define the left coset aH.
7. Given a subgroup $H \lhd G$, define the quotient group G/H. (Describe the set and the group operation.)
8. Given a permutation group $G < S_n$ acting on the set $\{1, 2, \ldots, n\}$, define the stabilizer $\mathrm{Stab}_G(i)$.
9. Given an element $a \in G$, define the centralizer $C(a)$.
10. Given $H < G$, define what it means for H to be normal in G.

11. Given two groups G_1 and G_2, define what it means for a map ϕ to be a homomorphism from G_1 to G_2.
12. Given a homomorphism $\phi : G_1 \to G_2$, define the kernel $\operatorname{Ker}\phi$.
13. Define the group of units $U(n)$.
14. Given the set $S = \{1, 2, \ldots, n\}$, define a permutation on S and the symmetric group S_n.
15. Given $a \in G$, define the conjugacy class $\mathrm{cl}(a)$.

2) Fill in the blanks or answer True/False (five from this list)

1. True or False: $(1234)(467) \in A_7$. __
2. How many elements of order 2 does D_4 have? __
3. True or False: $\langle(23)\rangle$ is a normal subgroup of S_3. __
4. True or False: If a and b are distinct elements of G and H is a subgroup of G, then the left cosets aH and bH are disjoint. __
5. $\mathbb{Z}_8/\langle 2\rangle$ is isomorphic to what well-known group? __
6. True or False: A consequence of Lagrange's Theorem is that any group of order 12 must contain an element of order 6. __
7. True or False: $U(8)$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. __
8. The order of the group $U(13)$ is __.
9. If $x^6 = x^{21} = e$ and $x \neq e$, then $|x| =$ __.
10. True or False: If $|G|$ is divisible by 11, then G has an subgroup of order 11. __
11. $11^{28} \equiv$ __ (mod 29).
12. True or False: For every positive integer n, $\mathrm{Aut}(U(n)) \approx \mathbb{Z}_n$. __
13. True or False: Let G be a cyclic group of order n. If $d|n$, then there is an $H < G$ of order d. __

14. True or False: $\mathbb{Z}_4 \oplus \mathbb{Z}_{10}$ is cyclic. __

15. True or False: Given an onto homomorphism $\phi : G_1 \to G_2$, it is true that $G_1/(\mathrm{Ker}\,\phi) \approx G_2$. __

3) Subgroups (one of the following)

1. Define the center $Z(G)$ of a group G. Prove that $Z(G)$ is a subgroup of G and that $Z(G) \lhd G$.

2. Let H be a nonempty finite subset of a group G. Then H is a subgroup of G if H is closed under the operation of G.

3. Let $\phi : G \to H$ be a homomorphism. Prove that the kernel $\mathrm{Ker}\,\phi$ is a subgroup of G.

4) Lagrange's Theorem (one of the following)

1. Use Lagrange's Theorem to prove: If G is a finite group and $x \in G$, then the order of x divides the order of G.

2. Use Lagrange's Theorem to prove Fermat's Little Theorem: for every integer a and every prime p, $a^p \equiv a \pmod{p}$.

3. The converse of Lagrange's Theorem is false. The group A_4 has order 12. Prove it has no subgroup of order 6.

5) Cosets (two of the following)

1. List the distinct left cosets of $H = \langle (13) \rangle$ in S_3.

2. Prove that A_n is a normal subgroup of S_n.

3. $(\mathbb{Z}_6 \oplus \mathbb{Z}_2)/\langle (2,1) \rangle$ is isomorphic to what group? Explain.

4. Given a subgroup $H < G$ and any $a, b \in G$, prove that either $aH = bH$ or aH and bH are disjoint.

5. Given a subgroup $H < G$ and any $a \in G$, prove that $aH < G$ iff $a \in H$.

6) Homomorphisms. Let $\phi : G_1 \to G_2$ be a homomorphism, and let $H < G$. (two of the following)

1. If $g \in G_1$ has finite order, then the order of $\phi(g)$ divides the order of g.
2. $\phi(H)$ is a subgroup of G_2.
3. If $H \lhd G_1$ then $\phi(H) \lhd \phi(G_1)$.
4. If $K < G_2$ then $\phi^{-1}(K) < G_1$.

7) First Isomorphism Theorem. Let $\phi : G \to H$ be a homomorphism of groups and let $K = \operatorname{Ker} \phi$. Let $\psi : G/K \to H$ be the correspondence that sends $gK \mapsto \phi(g)$. (two of the following)

1. Accurately state the First Isomorphism Theorem (also known as the Fundamental Theorem of Group Homomorphisms).
2. Prove that if $K = \{e\}$, then ϕ is one-to-one.
3. Show that ψ is well-defined. Prove that for any $x, y \in G$ such that $xK = yK$, we have $\phi(x) = \phi(y)$.
4. Show that ψ is one-to-one. Prove that for any $x, y \in G$ such that $\phi(x) = \phi(y)$, we have $xK = yK$.
5. Let H be the subgroup of $\mathrm{GL}(2, \mathbb{R})$ consisting of matrices of determinant 1. Use the First Isomorphism Theorem to prove that $\mathrm{GL}(2, \mathbb{R})/H \approx \mathbb{R}^*$ (where $\mathbb{R}^*$ is the group of non-zero reals under multiplication).

8) Euclidean Algorithm

1. Use the Euclidean Algorithm to express $\gcd(11, 28)$ as an integer linear combination of 11 and 28. Show all work.
2. Find the inverse of 11 in $U(28)$.

9) Cyclic groups (one of the following)

1. Let a be an element of a group G. Suppose that a has infinite order. Explain why $a^i = a^j$ implies that $i = j$.
2. Prove that if $|a| = k$, then $\langle a \rangle = \{e, a, a^2, \ldots, a^{k-1}\}$.

10-11) Two questions, each worth 10 points, will be randomly chosen from homework problems.