MA441: Algebraic Structures I

Lecture 1

3 September 2003

(From Chapter 2, Groups)

Definition:
Let G be a set. A **binary operation** on G is a function that assigns to each ordered pair of elements of G an element of G.

We can denote a binary operation f as
$f : G \times G \to G$.

We say that G is **closed** under a binary operation because the operation assigns an element of G to a pair, and not some element outside G.

(This property of a binary operation is called **closure**.)

Examples

The integers $\mathbb{Z}$ are closed under the binary operations of addition, subtraction and multiplication.

The integers are not closed under division.

Definition:
Let G be a nonempty set together with a binary operation (usually called multiplication) that assigns to each ordered pair (a, b) of elements of G an element of G denoted ab.

We say G is a **group** if the following three properties are satisfied:

1. **Associativity.** The operation is associative, that is, $(ab)c = a(bc)$, for all a, b, c in G.

2. **Identity.** There is an element e (called the **identity**) in G such that $ae = ea = a$ for all a in G.

3. **Inverses.** For each element a in G, there is an element b in G (called an **inverse** of a) such that $ab = ba = e$.

In other words, a group is a set together with an associative operation (sometimes called a **composition** rule) such that there is an identity, every element has an inverse, and any pair of elements can be combined without going outside the set.

The four properties to test in order to decide whether a set with an operation is a group are therefore closure, associativity, existence of an identity, and existence of inverses.

Definition:
If a group has the property that $ab = ba$ for every a, b in G, then the group is **Abelian**.

A group is **non-Abelian** if there is some pair of elements a, b for which $ab \neq ba$.

Example

The **dihedral group of order** $2n$, denoted D_n, is the group of symmetries of a regular n-gon.

The symmetries can be represented by rotations (denoted by R, or R_n for a rotation of n degrees) and flips (written F), which are reflections about an axis.