Selected Homework Solutions

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Chapter 3

3.6: We have that $x^2 \neq e$ and $x^6 = e$.

By way of contradiction, assume that $x^4 = e$. Then $e = x^6 x^{-4} = x^2$, which contradicts our assumption that x^2 is not the identity. Therefore $x^4 \neq e$.

By way of contradiction, assume that $x^5 = e$. Then $e = x^6 x^{-5} = x$. If $x = e$, then $x^2 = e$, which contradicts our assumption. Therefore $x^5 \neq e$.

Clearly the order of x can't be larger than 6 since $x^6 = e$. Since none of x, x^2, x^4, x^5 are the identity, the order of x must be either 3 or 6. If $|x| = 3$, then that satisfies the requirement that $x^6 = e$. The order of x may be 6 since we have $x^6 = e$. We don't have enough information to determine whether the order of x is 3 or 6.

3.14: Given $H, K < G$, show that $H \cap K < G$. We can use the One-step Subgroup test here. Suppose a, b are any elements of the intersection. Then $a, b \in H$ and $a, b \in K$. Because H and K are each subgroups, we know ab^{-1} is in both H and K, therefore in $H \cap K$.

Chapter 5

5.19: Given $H < S_n$, we need to show that either all elements of H are even or else that exactly half are even and half are odd.

Gallian's hint suggests to mimic the proof of Theorem 5.7. (I chose to right multiply by α instead of left multiply, but the idea is the same.)

If all the elements of H are even, then we are done. Otherwise, there is at least one odd element of H, so let $\alpha \in H$ be odd.

Let T be the map that right multiplies by α, that is, $T(x) = x\alpha$. Since H is a subgroup, multiplication by α sends an element of H to H.

Write H as the disjoint union of $H_e \cup H_o$ where H_e and H_o are respectively the subsets of even and odd permutations of H.

Note that T is a one-to-one map because α has an inverse. If $T(x) = T(y)$, then $x\alpha = y\alpha$ so $x = y$, because you can right multiply by α^{-1}.

$T(H_e) \subseteq H_o$ and $|H_e| \leq |H_o|$ because an even permutation times α, an odd permutation, is odd. Since T is one-to-one, there are at least as many odd permutations as even ones.

Conversely, $T(H_o) \subseteq H_e$, since an odd permutation times α, an odd permutation, is even. Since T is one-to-one, there are at least as many even permutations as odd ones.

Therefore the number of even permutations in H is equal to the number of odd permutations in H.

Another approach is to argue that T is a bijection. We showed that T is one-to-one. Note that α^{-1} is odd as well, because when you write α as a product of 2-cycles, its inverse can be written with the same number of 2-cycles. Given any odd permutation $\gamma \in H_o$, we form an even permutation by right multiplying by α^{-1}. A preimage for γ under T is $\gamma\alpha^{-1}$, since $T(\gamma\alpha^{-1}) = \gamma\alpha^{-1}\alpha = \gamma$. Since T is one-to-one and onto, it is a bijection and therefore $|H_e| = |H_o|$.

Chapter 6

6.6 Let $\phi : G \to H$ and $\psi : H \to K$ be isomorphisms. We want to show the composition $\psi \circ \phi : G \to K$ is an isomorphism.

Since both ϕ and ψ are one-to-one and onto, their composition $\psi \circ \phi$ is also one-to-one and onto.

To show that the composition preserves the group operation, take any $a, b \in G$:

$$(\psi \circ \phi)(a \cdot b) = \psi(\phi(a) \cdot \phi(b)) = \psi(\phi(a)) \cdot \psi(\phi(b)) = (\psi \circ \phi)(a) \cdot (\psi \circ \phi)(b).$$

6.32 The inner automorphism ϕ_g sends x to gxg^{-1}. Similarly, $\phi_{zg}(x) = (zg)x(zg)^{-1} = zgxg^{-1}z^{-1}$. We have that $z \in Z(G)$. Therefore z commutes with all elements, including g, x, g^{-1}. So $zgxg^{-1}z^{-1} = gxg^{-1}zz^{-1} = gxg^{-1} = \phi_g(x)$.

6.35 Let $|a| = n$. To show $|\phi_a|$ divides n, consider composing ϕ_a with itelf. $(\phi_a)^2(x) = a(axa^{-1})a^{-1} = a^2xa^{-2}$. Similarly $(\phi_a)^n(x) = a^nxa^{-n}$, which equals x since $a^n = e$. Because $(\phi_a)^n$ is the identity map, its order must divide n.

Consider D_4, and let $a = R_{90}$. Then $|a| = 4$, and we will show $|\phi_a| = 2$.

Write the elements of D_4 in terms of a flip F and a rotation $R = R_{90}$. One can verify by inspection that the map $\phi_a(x) = RxR^{-1}$ has order 2, that is, $(\phi_a)^2(x) = x$.

There are better ways to show this than simply by checking all cases. Since every element of D_4 can be written as F^iR^j, where i is either 0 or 1 and $j \in \{0, 1, 2, 3\}$, it suffices to verify that $(\phi_a)^2(F) = F$ and $(\phi_a)^2(R) = R$ since ϕ_a is a homomorphism: $\phi_a(F^iR^j) = \phi(F)^i\phi(R)^j$.

Alternatively, we can use the relation $RF = FR^{-1}$. Then

$$(\phi_a)^2(F^iR^j) = R^2F^iR^jR^{-2} = F^iR^{-2}R^{j-2} = F^iR^{j-4} = F^iR^j,$$

which shows that ϕ_a^2 is the identity.

Chapter 8

8.11 By Theorem 8.2, $\mathbb{Z}_3 \oplus \mathbb{Z}_5$ is a cyclic group. It has 15 elements because it is a direct product of groups of orders 3 and 5. It is isomorphic to $\mathbb{Z}_{15}$ because this group is also cyclic of order 15. In fact, the map sending $(1, 1) \mapsto 1$ gives the isomorphism.

Chapter 9

9.3 The alternating group A_n is a normal subgroup of S_n because it has index 2. Let α be an odd permutation in S_n. Then S_n can be partitioned into the two left cosets A_n and αA_n. The subset of odd permutations is equal to αA_n. We can also partition S_n into right cosets A_n and $A_n\alpha$. The subset of odd permutations is equal to $A_n\alpha$. Since $A_n\alpha$ and αA_n both equal the set of odd permutations, they are equal. Obviously this relation also holds if $\alpha \in A_n$, since A_n is a subgroup. Since $\alpha A_n = A_n\alpha$ for any α, we have $A_n \lhd S_n$.